ASYMPTOTICS FOR RANK PARTITION FUNCTIONS

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ABSTRACT. In this paper, we obtain asymptotic formulas for an infinite class of rank generating functions. As an application, we solve a conjecture of Andrews and Lewis on inequalities between certain ranks.

1. INTRODUCTION AND STATEMENT OF RESULTS

A *partition* of a positive integer n is any non-increasing sequence of positive integers whose sum is n. As usual, let $p(n)$ denote the number of partitions of n. The partition function $p(n)$ has the well known infinite product generating function

$$(1.1) \qquad 1 + \sum_{n=1}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1-q)^2(1-q^2)^2 \cdots (1-q^n)^2}.$$

Hardy and Ramanujan showed the following asymptotic formula for $p(n)$

$$p(n) \sim \frac{1}{4n\sqrt{3}} \cdot e^{\pi\sqrt{2n/3}} \qquad (n \to \infty).$$

Using the modularity of the generating function for $p(n)$, Rademacher obtained an exact formula for $p(n)$. To state his result, let $I_s(x)$ be the usual I-Bessel function of order s, and let $e(x) := e^{2\pi i x}$. Furthermore, if $k \geq 1$ and n are integers, then let

$$(1.2) \qquad A_k(n) := \frac{1}{2}\sqrt{\frac{k}{12}} \sum_{\substack{x \pmod{24k} \\ x^2 \equiv -24n+1 \pmod{24k}}} \chi_{12}(x) \cdot e\left(\frac{x}{12k}\right),$$

where the sum runs over the residue classes modulo $24k$, and where

$$(1.3) \qquad \chi_{12}(x) := \left(\frac{12}{x}\right).$$

If n is a positive integer, then Rademacher showed that

$$(1.4) \qquad p(n) = \frac{2\pi}{(24n-1)^{3/4}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} \cdot I_{\frac{3}{2}}\left(\frac{\pi\sqrt{24n-1}}{6k}\right).$$

The partition function also satisfies some nice congruence properties; the most famous ones are the so-called Ramanujan congruences:

$$p(5n + 4) \equiv 0 \pmod 5,$$
$$p(7n + 5) \equiv 0 \pmod 7,$$
$$p(11n + 6) \equiv 0 \pmod{11}.$$

In order to explain the congruences with modulus 5 and 7 combinatorially, Dyson [9] introduced the "rank" of a partition. The rank of a partition is defined to be its largest part minus the number of its parts. Dyson conjectured that the partitions of $5n+4$ (resp. $7n+5$) form 5 (resp. 7) groups of equal size when sorted by their ranks modulo 5 (resp. 7). This conjecture was proved in 1954 by Atkin and Swinnerton-Dyer [5].

If $N(m,n)$ denotes the number of partitions of n with rank m, then it is well known that

$$(1.5) \qquad R(w;q) := 1 + \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} N(m,n) w^m q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(wq;q)_n (w^{-1}q;q)_n},$$

where

$$(a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

Obviously, by letting $w = 1$, we obtain (1.1). Moreover, if $N_e(n)$ (resp. $N_o(n)$) denotes the number of partitions of n with even (resp. odd) rank, then by letting $w = -1$ we obtain

$$(1.6) \qquad 1 + \sum_{n=1}^{\infty} (N_e(n) - N_o(n)) q^n = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2 (1+q^2)^2 \cdots (1+q^n)^2}.$$

In the following we denote this series by $f(q)$ and its n-th Fourier coefficient by $\alpha(n)$. The series $f(q)$ is one of the third order mock theta functions defined by Ramanujan in his last letter to Hardy dated January 1920 (see pages 127-131 of [14]). There Ramanujan claimed, without including a proof, that

$$\alpha(n) = (-1)^{n-1} \frac{\exp\left(\pi\sqrt{\frac{n}{6} - \frac{1}{144}}\right)}{2\sqrt{n - \frac{1}{24}}} + O\left(\frac{\exp\left(\frac{1}{2}\pi\sqrt{\frac{n}{6} - \frac{1}{144}}\right)}{\sqrt{n - \frac{1}{24}}}\right).$$

Dragonette [8] proved this claim in her Ph.D. thesis written 1951 under the direction of Rademacher. Andrews [2] improved this in his Ph.D. thesis 1964, also written under Rademacher, as

$$(1.7) \quad \alpha(n) = \frac{\pi}{\sqrt{24n-1}} \sum_{k=1}^{[\sqrt{n}]} \frac{(-1)^{\lfloor \frac{k+1}{2} \rfloor} A_{2k}\left(n - \frac{k(1+(-1)^k)}{4}\right)}{k} \cdot I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12k}\right) + O(n^\epsilon).$$

Moreover Andrews and Dragonette made the following conjecture.

Conjecture. (Andrews-Dragonette)
If n is a positive integer, then

$$(1.8) \qquad \alpha(n) = \frac{\pi}{(24n-1)^{\frac{1}{4}}} \sum_{k=1}^{\infty} \frac{(-1)^{\lfloor \frac{k+1}{2} \rfloor} A_{2k}\left(n - \frac{k(1+(-1)^k)}{4}\right)}{k} \cdot I_{\frac{1}{2}}\left(\frac{\pi\sqrt{24n-1}}{12k}\right).$$

In [6] Ono and the author proved this conjecture using the theory of Maass-Poincaré series. It turns out that $q^{-1}f\left(q^{24}\right)$ is the "holomorphic part" of a weak Maass form (see [7] for the definition of a weak Maass form). In [7] they showed that a similar phenomenon is true for all functions

$$R(\zeta_c^a; q) =: 1 + \sum_{n=1}^{\infty} A\left(\frac{a}{c}; n\right) q^n,$$

where $\zeta_n := e^{\frac{2\pi i}{n}}$ and $0 < a < c$ are integers. More specificly, $R\left(\zeta_c^a; q\right)$ is the holomorphic part of a weak Maass form of weight $\frac{1}{2}$. Using this deeper insight, in this paper, we are able to obtain asymptotic formulas for all the coefficients $A\left(\frac{a}{c}; n\right)$, which in turn implies asymptotics for the rank partition functions.

Before we state our result, we need some more notation. We let k and h be coprime integers, h' defined by $hh' \equiv -1 \pmod{k}$ if k is odd and $hh' \equiv -1 \pmod{2k}$ if k is even, $k_1 := \frac{k}{\gcd(k,c)}$, $c_1 := \frac{c}{\gcd(k,c)}$ and $0 < l < c_1$ defined by the congruence $l \equiv ak_1 \pmod{c_1}$. If $\frac{b}{c} \in (0,1) \setminus \{\frac{1}{2}, \frac{1}{6}, \frac{5}{6}\}$, then define the integer $s(b,c)$ by

$$(1.9) \qquad s(b,c) := \begin{cases} 0 & \text{if } 0 < \frac{b}{c} < \frac{1}{6}, \\ 1 & \text{if } \frac{1}{6} < \frac{b}{c} < \frac{1}{2}, \\ 2 & \text{if } \frac{1}{2} < \frac{b}{c} < \frac{5}{6}, \\ 3 & \text{if } \frac{5}{6} < \frac{b}{c} < 1. \end{cases}$$

In particular, set $s := s(l, c_1)$. Let $\omega_{h,k}$ be the multiplier occuring in the transformation law of the partition function $p(n)$. This is explicitly given by

$$(1.10) \qquad \omega_{h,k} := \exp\left(\pi i t(h,k)\right),$$

where

$$t(h,k) := \sum_{\mu \pmod k} \left(\left(\frac{\mu}{k}\right)\right)\left(\left(\frac{h\mu}{k}\right)\right).$$

Here

$$((x)) := \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

Moreover we define, for $n, m \in \mathbb{Z}$, the following sums of Kloosterman type

$$(1.11) \quad B_{a,c,k}(n,m) := (-1)^{ak+1} \sin\left(\frac{\pi a}{c}\right) \sum_{h \pmod k^*} \frac{\omega_{h,k}}{\sin\left(\frac{\pi a h'}{c}\right)} \cdot e^{-\frac{3\pi i a^2 k_1 h'}{c}} \cdot e^{\frac{2\pi i}{k}(nh+mh')}$$

if $c|k$, and

$$(1.12) \qquad D_{a,c,k}(n,m) := (-1)^{ak+l} \sum_{h \pmod{k}^*} \omega_{h,k} \cdot e^{\frac{2\pi i}{k}(nh+mh')}.$$

Here the sums run through all primitive residue classes modulo k. Moreover, for $c \nmid k$, let

$$(1.13) \qquad \delta_{c,k,r} := \begin{cases} -\left(\frac{1}{2}+r\right)\frac{l}{c_1} + \frac{3}{2}\left(\frac{l}{c_1}\right)^2 + \frac{1}{24} & \text{if } 0 < \frac{l}{c_1} < \frac{1}{6}, \\ -\frac{5l}{2c_1} + \frac{3}{2}\left(\frac{l}{c_1}\right)^2 + \frac{25}{24} - r\left(1 - \frac{l}{c_1}\right) & \text{if } \frac{5}{6} < \frac{l}{c_1} < 1, \\ 0 & \text{otherwise,} \end{cases}$$

and for $0 < \frac{l}{c_1} < \frac{1}{6}$ or $\frac{5}{6} < \frac{l}{c_1} < 1$

$$m_{a,c,k,r} := \begin{cases} \frac{1}{2c_1^2}\left(-3a^2 k_1^2 + 6lak_1 - ak_1 c_1 - 3l^2 + lc_1 - 2ark_1 c_1 + 2lc_1 r\right) & \text{if } 0 < \frac{l}{c_1} < \frac{1}{6}, \\ \frac{1}{2c_1^2}\left(-6ak_1 c_1 - 3a^2 k_1^2 + 6lak_1 + ak_1 c_1 + 6lc_1 \right. & \text{if } \frac{5}{6} < \frac{l}{c_1} < 1. \\ \left. -3l^2 - 2c_1^2 - lc_1 + 2ark_1 c_1 + 2c_1(c_1 - l)r\right) \end{cases}$$

Remark. It is not hard to see that $m_{a,c,k,r} \in \mathbb{Z}$.

We obtain following asymptotic formulas for the coefficients $A\left(\frac{a}{c};n\right)$.

Theorem 1.1. *If $0 < a < c$ are coprime integers and c is odd, then for positive integers n we have that*

$$A\left(\frac{a}{c};n\right) = \frac{4\sqrt{3}i}{\sqrt{24n-1}} \sum_{\substack{1 \le k \le \sqrt{n} \\ c|k}} \frac{B_{a,c,k}(-n,0)}{\sqrt{k}} \cdot \sinh\left(\frac{\pi\sqrt{24n-1}}{6k}\right)$$

$$+ \frac{8\sqrt{3} \cdot \sin\left(\frac{\pi a}{c}\right)}{\sqrt{24n-1}} \sum_{\substack{1 \le k \le \sqrt{n} \\ c \nmid k \\ r \ge 0 \\ \delta_{c,k,r} > 0}} \frac{D_{a,c,k}(-n, m_{a,c,k,r})}{\sqrt{k}} \cdot \sinh\left(\frac{\pi\sqrt{2\delta_{c,k,r}(24n-1)}}{\sqrt{3}k}\right) + O_c\left(n^\epsilon\right).$$

Four remarks.
1) One can easily see that the second sum is empty for $c \in \{3,5\}$.
2) For a fixed choice of a and c, we can in principle obtain exact formulas for $A\left(\frac{a}{c};n\right)$ by modifying an argument given in [6] to prove (1.8).
3) Similarly as in this paper, one can also prove asymptotic formulas for generalized ranks as defined in [11]. Since the proof is basically the same, we do not give it here.
4) One could also generalize our results to the case that c is even, but for simplicity we restrict us to the case that c is odd.

If we denote by $N(a,c;n)$ the number of partitions of n with rank congruent $a \pmod{c}$ then it is easy to conclude the following:

Corollary 1.2. *For integers $0 \le a < c$, where c is an odd integer, we have*

$$N(a, c; n) = \frac{2\pi}{c \cdot \sqrt{24n - 1}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} \cdot I_{\frac{3}{2}} \left(\frac{\pi \sqrt{24n - 1}}{6k} \right)$$

$$+ \frac{1}{c} \sum_{j=1}^{c-1} \zeta_c^{-aj} \left(\frac{4\sqrt{3}i}{\sqrt{24n - 1}} \sum_{c|k} \frac{B_{j,c,k}(-n, 0)}{\sqrt{k}} \sinh \left(\frac{\pi}{6k} \sqrt{24n - 1} \right) \right.$$

$$\left. + \frac{8\sqrt{3} \sin \left(\frac{\pi j}{c} \right)}{\sqrt{24n - 1}} \sum_{\substack{k, r \\ c \nmid k \\ \delta_{c,k,r} > 0}} \frac{D_{j,c,k}(-n, m_{j,c,k,r})}{\sqrt{k}} \sinh \left(\sqrt{\frac{2\delta_{c,k,r}(24n - 1)}{3}} \frac{\pi}{k} \right) \right) + O_c \left(n^\epsilon \right).$$

This corollary implies some conjectures of Andrews and Lewis. In [4, 13] they showed

$$\begin{array}{llll}
N(0, 2; 2n) & < & N(1, 2; 2n) & \text{if } n \ge 1, \\
N(0, 4; n) & > & N(2, 4; n) & \text{if } 26 < n \equiv 0, 1 \pmod{4}, \\
N(0, 4; n) & < & N(2, 4; n) & \text{if } 26 < n \equiv 2, 3 \pmod{4}.
\end{array}$$

Moreover, they conjectured (see Conjecture 1 of [4]).

Conjecture. (Andrews and Lewis)
For all $n > 0$, we have

(1.14)
$$\begin{array}{ll}
N(0, 3; n) < N(1, 3; n) & \text{if } n \equiv 0 \text{ or } 2 \pmod{3}, \\
N(0, 3; n) > N(1, 3; n) & \text{if } n \equiv 1 \pmod{3},
\end{array}$$

A careful analysis of Corollary 1.2 gives the following theorem.

Theorem 1.3. *The Andrews-Lewis Conjecture is true for all $n \notin \{3, 9, 21\}$ in which case we have equality in (1.14).*

Remark. From Corollary 1.2 we see that

(1.15)
$$N(0, 3; n) - N(1, 3; n) \sim - \frac{8 \sin \left(\frac{\pi}{18} - \frac{2\pi n}{3} \right) \sinh \left(\frac{\pi \cdot \sqrt{24n-1}}{18} \right)}{\sqrt{24n - 1}}.$$

This directly implies the Andrews-Lewis Conjecture for n sufficiently large n. For example for $n = 1200$, we have

$$N(0, 3; 1200) - N(1, 3; 1200) = -29873238925$$

whereas (1.15) gives

$$-29873204830.34.$$

The paper is organized as follows: In Section 2 we prove a transformation law for the functions $R(\zeta_c^a; q)$. The behavior under the generators of $\mathrm{SL}_2(\mathbb{Z})$ was also studied in [12] and in [7]. However here we prove a more general result since we need the occurring roots of unity and integrals explicitly for every element in $\mathrm{SL}_2(\mathbb{Z})$. In Section 4 we prove Theorem 1.1 and

Corollary 1.2 by using the Circle Method. For this we need some estimates shown in Section 3. Section 5 is dedicated to the proof of Theorem 1.3.

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2. MODULAR TRANSFORMATION FORMULAS

In this section we prove a transformation law for the functions $R\left(\zeta_c^a; q\right)$. For this define

$$(2.1) \quad N\left(\frac{a}{c}; q\right) := \frac{1}{(q;q)_\infty}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n\left(1+q^n\right)\left(2 - 2\cos\left(\frac{2\pi a}{c}\right)\right)}{1 - 2q^n\cos\left(\frac{2\pi a}{c}\right) + q^{2n}} \cdot q^{\frac{n(3n+1)}{2}}\right).$$

In [12] it is shown that

$$R\left(\zeta_c^a; q\right) = N\left(\frac{a}{c}; q\right).$$

Moreover let

$$N(a,b,c;q) := \frac{i}{2(q;q)_\infty}\left(\sum_{m=0}^{\infty} \frac{(-1)^m e^{-\frac{\pi i a}{c}} \cdot q^{\frac{m}{2}(3m+1)+ms(b,c)+\frac{b}{2c}}}{1 - e^{-\frac{2\pi i a}{c}} \cdot q^{m+\frac{b}{c}}}\right.$$
$$\left. - \sum_{m=1}^{\infty} \frac{(-1)^m e^{\frac{\pi i a}{c}} \cdot q^{\frac{m}{2}(3m+1)-ms(b,c)-\frac{b}{2c}}}{1 - e^{\frac{2\pi i a}{c}} \cdot q^{m-\frac{b}{c}}}\right).$$

Remark. It is easy to see that the above definition coincides with the definition, given in [7].

For each $\nu \in \mathbb{Z}$ define

$$H_{a,c}(x) := \frac{\cosh(x)}{\sinh\left(x + \frac{\pi i a}{c}\right) \cdot \sinh\left(x - \frac{\pi i a}{c}\right)},$$

$$I_{a,c,k,\nu}(z) := \int_{\mathbb{R}} e^{-\frac{3\pi z x^2}{k}} \cdot H_{a,c}\left(\frac{\pi i \nu}{k} - \frac{\pi i}{6k} - \frac{\pi z x}{k}\right) dx.$$

It is easy to see that

$$H_{a,c}(-x) = H_{a,c}(x)$$

$$H_{a,c}(x) = \frac{(1+q) \cdot q^{\frac{1}{2}}}{2\left(1 - 2\cos\left(\frac{2\pi a}{c}\right)q + q^2\right)}$$

for $q = e^{2x}$. We show the following transformation law for the function $N\left(\frac{a}{c}; q\right)$.

Theorem 2.1. *We assume the same notation as in the introduction. Moreover, let $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$, $q := e^{\frac{2\pi i}{k}(h+iz)}$, and $q_1 := e^{\frac{2\pi i}{k}\left(h'+\frac{i}{z}\right)}$.*

(1) *If $c|k$, then*

$$N\left(\frac{a}{c};q\right) = \frac{(-1)^{ak+1}i\sin\left(\frac{\pi a}{c}\right)\cdot\omega_{h,k}}{\sin\left(\frac{\pi ah'}{c}\right)z^{\frac{1}{2}}}\cdot e^{-\frac{3\pi ia^2 k_1 h'}{c}}\cdot e^{\frac{\pi}{12k}\left(z^{-1}-z\right)}\cdot N\left(\frac{ah'}{c};q_1\right)$$

$$+\frac{2\sin^2\left(\frac{\pi a}{c}\right)\cdot\omega_{h,k}}{k}e^{-\frac{\pi z}{12k}}\cdot z^{\frac{1}{2}}\sum_{\nu\pmod k}(-1)^\nu e^{-\frac{3\pi ih'\nu^2}{k}+\frac{\pi ih'\nu}{k}}\cdot I_{a,c,k,\nu}(z).$$

(2) *If $c\nmid k$, then*

$$N\left(\frac{a}{c};q\right) = \frac{4i(-1)^{ak+l+1}\cdot\sin\left(\frac{\pi a}{c}\right)\omega_{h,k}}{z^{\frac{1}{2}}}\cdot e^{-\frac{2\pi ih'sa}{c}-\frac{3\pi ia^2 h'k_1}{cc_1}+\frac{6\pi ih'la}{cc_1}}\cdot q_1^{\frac{sl}{c_1}-\frac{3l^2}{2c_1^2}}\cdot e^{\frac{\pi}{12k}\left(z^{-1}-z\right)}$$

$$\times N\left(ah',\frac{lc}{c_1},c;q_1\right)+\frac{2\sin^2\left(\frac{\pi a}{c}\right)\omega_{h,k}}{k}e^{-\frac{\pi z}{12k}}\cdot z^{\frac{1}{2}}\sum_{\nu\pmod k}(-1)^\nu e^{-\frac{3\pi ih'\nu^2}{k}+\frac{\pi ih'\nu}{k}}I_{a,c,k,\nu}(z).$$

Proof. We modify the proof of [2]. We easily see, using $1-\cos(2x)=2\sin(x)^2$ that

$$(2.2)\quad (q;q)_\infty\cdot N\left(\frac{a}{c};q\right)=\sin^2\left(\frac{\pi a}{c}\right)\sum_{n\in\mathbb{Z}}(-1)^n\ H_{a,c}\left(\frac{\pi in}{k}(h+iz)\right)\cdot e^{\frac{3\pi i(h+iz)n^2}{k}}.$$

Writing $n=km+\nu$ with $0\leq\nu<k$, $m\in\mathbb{Z}$ and using that $(h,k)=1$, gives that (2.2) equals

$$(2.3)\quad \sin^2\left(\frac{\pi a}{c}\right)\sum_{\nu=0}^{k-1}(-1)^\nu e^{\frac{3\pi ih\nu^2}{k}}\sum_{m\in\mathbb{Z}}(-1)^m\ H_{a,c}\left(\frac{\pi ih\nu}{k}-\frac{\pi(km+\nu)z}{k}\right)\cdot e^{-\frac{3\pi z(km+\nu)^2}{k}}.$$

Using Poisson summation and substituting $x\mapsto kx+\nu$ gives that the inner sum equals

$$(2.4)\qquad \frac{1}{k}\sum_{n\in\mathbb{Z}}\int_\mathbb{R}H_{a,c}\left(\frac{\pi ih\nu}{k}-\frac{\pi xz}{k}\right)\cdot e^{\frac{\pi i(2n+1)(x-\nu)}{k}-\frac{3\pi zx^2}{k}}dx.$$

Strictly speaking for $c|k$ there may be a pole at $x=0$. In this case we take the principal part of the integral. Inserting (2.4) into (2.3) we see that the summation only depends on ν (mod k). Moreover, by changing ν into $-\nu$, x into $-x$, and n into $-(n+1)$, we see that the sum over n with $n\leq-1$ equals the sum with $n\geq0$. Thus (2.3) equals

$$(2.5)\quad \frac{2\sin^2\left(\frac{\pi a}{c}\right)}{k}\sum_{\nu\pmod k}(-1)^\nu e^{\frac{3\pi ih\nu^2}{k}}\sum_{n\in\mathbb{N}}\int_\mathbb{R}H_{a,c}\left(\frac{\pi ih\nu}{k}-\frac{\pi xz}{k}\right)\cdot e^{\frac{\pi i(2n+1)(x-\nu)}{k}-\frac{3\pi zx^2}{k}}dx.$$

To see where the poles of the integrant lie, we introduce the function

$$S_{a,c,k}(x):=\frac{\sinh(c_1 x)}{\sinh\left(\frac{x}{k}+\frac{\pi ia}{c}\right)\cdot\sinh\left(\frac{x}{k}-\frac{\pi ia}{c}\right)}$$

which is entire as a function of x. Using that c is odd we can write the integrand in (2.5) as

$$\frac{(-1)^{h\nu}\cosh\left(\frac{\pi ih\nu}{k}-\frac{\pi xz}{k}\right)\cdot e^{\frac{\pi i(2n+1)(x-\nu)}{k}-\frac{3\pi zx^2}{k}}\cdot S_{a,c,k}\left(\pi xz-\pi ih\nu\right)}{\sinh(\pi c_1 xz)}.$$

From this we see that the only poles can lie in the points

$$x_m := \frac{im}{c_1 z} \qquad (m \in \mathbb{Z}).$$

We treat the cases with $c | k$ or $c \nmid k$ seperately.

If $c | k$, then $c_1 = 1$. One computes that each choice $\pm$ leads at most for one $\nu \pmod{k}$ to a non-zero residue, and that this ν can for $\epsilon \in \{\pm\}$ be chosen as

$$\nu_m^\epsilon := -h'(m - \epsilon a k_1).$$

We denote the corresponding residues by $\lambda_{n,m}^\epsilon$ ($\epsilon \in \{\pm\}$). By shifting the path of integration through the points

$$\omega_n := \frac{(2n+1)i}{6z},$$

we have to take those points x_m into account for which $n \geq 3m \geq 0$. Setting $r_0 := \frac{1}{2}$ and $r_m := 1$ for $m \in \mathbb{N}$, we obtain by the Residue Theorem

$$(q;q)_\infty \cdot N\left(\frac{a}{c};q\right) = \sum_{11} + \sum_2,$$

where

$$\sum_{11} := \frac{4\pi i \sin^2\left(\frac{\pi a}{c}\right)}{k} \sum_{\substack{m \geq 0 \\ \epsilon \in \{\pm\}}} r_m (-1)^{\nu_m^\epsilon} e^{\frac{3\pi i h (\nu_m^\epsilon)^2}{k}} \sum_{n=3m}^\infty \lambda_{n,m}^\epsilon,$$

$$\sum_2 := \frac{2 \sin^2\left(\frac{\pi a}{c}\right)}{k} \sum_{\nu \pmod{k}} (-1)^\nu e^{\frac{3\pi i h \nu^2}{k}}$$

$$\sum_{n \in \mathbb{N}} \int_{-\infty+\omega_n}^{\infty+\omega_n} H_{a,c}\left(\frac{\pi i h \nu}{k} - \frac{\pi x z}{k}\right) \cdot e^{\frac{\pi i (2n+1)(x-\nu)}{k} - \frac{3\pi x^2 z}{k}} \, dx.$$

If $c \nmid k$, then a pole can only occur if $m \equiv \pm a k_1 \pmod{c_1}$. Writing $c_1 m \pm l$ instead of m with $m \geq 0$ for the choice $+$ and $m > 0$ for the choice $-$ and l as in the introduction, we see that to each choice there corresponds exactly one $\nu \pmod{k}$ and we can choose ν for $\epsilon \in \{\pm 1\}$ as

$$\nu_m^\epsilon := -h'\left(m \, \epsilon \frac{1}{c_1}(l - a k_1)\right).$$

As before, we denote the corresponding residues by $\lambda_{n,m}^\epsilon$. By shifting the path of integration through the points ω_n, we have to take those points x_m into account for which $\frac{2n+1}{6} > \frac{c_1 m \epsilon l}{c_1}$. One can see that this is equivalent to

$$n \geq 3m\epsilon s,$$

where s was defined in (1.9). By the Residue Theorem we obtain

$$(q;q)_\infty \cdot N\left(\frac{a}{c};q\right) = \sum_{12} + \sum_2,$$

where $\sum_2$ is given as before and $\sum_{12}$ is defined as

$$\frac{4\pi i \sin^2\left(\frac{\pi a}{c}\right)}{k}\left(\sum_{m\geq 0}(-1)^{\nu_m^+}e^{\frac{3\pi i h(\nu_m^+)^2}{k}}\sum_{n=3m+s}^{\infty}\lambda_{n,m}^+ + \sum_{m\geq 1}(-1)^{\nu_m^-}e^{\frac{3\pi i h(\nu_m^-)^2}{k}}\sum_{n=3m-s}^{\infty}\lambda_{n,m}^-\right).$$

We first consider the sums $\sum_{11}$ and $\sum_{12}$. We have

$$\lambda_{n,m}^\epsilon = -\frac{k\cdot\cosh\left(\frac{\pi i h\nu_m^\epsilon}{k}-\frac{\pi x_m z}{k}\right)\cdot e^{\frac{\pi i(2n+1)(x_m-\nu_m^\epsilon)}{k}-\frac{3\pi z x_m^2}{k}}}{\pi z\cdot\cosh\left(\frac{\pi i h\nu_m^\epsilon}{k}-\frac{\pi x_m z}{k}\epsilon\frac{\pi i a}{c}\right)\cdot\sinh\left(\frac{\pi i h\nu_m^\epsilon}{k}-\frac{\pi x_m z}{k}-\epsilon\frac{\pi i a}{c}\right)}.$$

From this one directly sees that

$$\lambda_{n+1,m}^\epsilon = \exp\left(\frac{2\pi i}{k}\left(x_m-\nu_m^\epsilon\right)\right)\cdot\lambda_{n,m}^\epsilon.$$

Thus

$$\sum_{n=3m+r}^{\infty}\lambda_{n,m}^\epsilon = \frac{\lambda_{3m+r,m}^\epsilon}{1-\exp\left(\frac{2\pi i}{k}\left(x_m-\nu_m^\epsilon\right)\right)}\qquad(r\in\{0,\pm s\},\epsilon\in\{\pm\}).$$

A straightforward but lenghty calculation gives

$$\sum_{11} = \frac{(-1)^{ak+1}i\sin\left(\frac{\pi a}{c}\right)}{\sin\left(\frac{\pi a h'}{c}\right)z}\cdot e^{-\frac{3\pi i a^2 k_1 h'}{c}}\cdot(q_1;q_1)_\infty\cdot N\left(\frac{ah'}{c};q_1\right),$$

$$\sum_{12} = \frac{4i\sin\left(\frac{\pi a}{c}\right)(-1)^{ak+l+1}}{z}e^{-\frac{2\pi i h' sa}{c}-\frac{3\pi i h' a^2 k_1}{cc_1}+\frac{6\pi i h' la}{cc_1}}q_1^{\frac{sl}{c_1}-\frac{3}{2}\left(\frac{l}{c_1}\right)^2}(q_1;q_1)_\infty N\left(h'a,\frac{lc}{c_1},c;q_1\right).$$

Here we have to be a little careful since for $c|k$ $(c\nmid k)$ we only have that $h\nu$ is congruent to $m-\epsilon a k_1$ $(m\epsilon\frac{1}{c_1}(l-ak_1))$ modulo k but not necessarily modulo $2k$ if k is odd.

We next turn to the computation of $\sum_2$. With the same argument as before we can change the sum over $\mathbb{N}$ into a sum over $\mathbb{Z}$. Making the translation $x\mapsto x+\omega_n$ and writing $n=3p+\delta$ with $p\in\mathbb{Z}$ and $\delta\in\{0,\pm 1\}$ gives that

$$\sum_2 = \frac{\sin^2\left(\frac{\pi a}{c}\right)}{k}\sum_{\nu\pmod k}(-1)^\nu e^{\frac{3\pi i h\nu^2}{k}}\sum_{\substack{p\in\mathbb{Z}\\\delta\in\{0,\pm 1\}}}e^{-\frac{\pi(6p+2\delta+1)^2}{12kz}-\frac{\pi i\nu(6p+2\delta+1)}{k}}$$

$$\int_{-\infty}^{\infty}H_{a,c}\left(\frac{\pi i h\nu}{k}-\frac{\pi i(6p+2\delta+1)}{6k}-\frac{\pi zx}{k}\right)\cdot e^{-\frac{3\pi z x^2}{k}}\,dx.$$

Now $(h', k) = 1$ implies that $-h'(\nu + p)$ runs modulo k if ν does. Thus we can change ν into $-h'(\nu + p)$ which leads to

$$(2.6) \quad \sum_2 = \frac{\sin^2\left(\frac{\pi a}{c}\right)}{k} \sum_{\substack{\nu \pmod{k} \\ \delta \in \{0, \pm 1\} \\ p \in \mathbb{Z}}} (-1)^{\nu + p} \, q_1^{\frac{p}{2}(3p + 2\delta + 1)} \cdot e^{-\frac{\pi(2\delta+1)^2}{12kz} + \frac{\pi i h'\left(-3\nu^2 + (2\delta+1)\nu\right)}{k}}$$

$$\int_{-\infty}^{\infty} H_{a,c}\left(\frac{\pi i \nu}{k} - \frac{\pi i (2\delta + 1)}{6k} - \frac{\pi z x}{k}\right) \cdot e^{-\frac{3\pi z x^2}{k}} \, dx.$$

Now the integral is independent of p and the sum over p equals

$$(2.7) \qquad\qquad \sum_{p \in \mathbb{Z}} (-1)^p \cdot q_1^{\frac{1}{2}p(3p + (2\delta+1))}.$$

If $\delta = 1$, then (2.7) vanishes since the p−th and the $-(p+1)$-th term cancel. Changing ν into $-\nu$, p into $-p$ and x into $-x$, we see that the terms in (2.6) corresponding to $\delta = -1$ and $\delta = 0$ are equal. Moreover in this case (2.7) equals $(q_1; q_1)_\infty$. Thus

$$\sum_2 = \frac{2\sin^2\left(\frac{\pi a}{c}\right) \cdot (q_1; q_1)_\infty \cdot e^{-\frac{\pi}{12kz}}}{k} \sum_{\nu \pmod{k}} (-1)^\nu e^{-\frac{3\pi i h' \nu^2}{k} + \frac{\pi i h' \nu}{k}} \cdot I_{a,c,k,\nu}(z).$$

Now the theorem follows easily using the transformation law

$$(q_1; q_1)_\infty = \omega_{h,k} \cdot z^{\frac{1}{2}} \cdot e^{\frac{\pi}{12k}\left(z^{-1} - z\right)} \cdot (q; q)_\infty.$$

$\square$

3. Some estimates

In this section we estimate the function $I_{a,c,k,\nu}(z)$, defined in Section 2, and some Kloosterman sums.

Lemma 3.1. *Assume that* $n \in \mathbb{N}$, $\nu \in \mathbb{Z}$, $z := \frac{k}{n} - k\Phi i$, $-\frac{1}{k(k+k_1)} \leq \Phi \leq \frac{1}{k(k+k_2)}$, *where* $\frac{h_1}{k_1} < \frac{h}{k} < \frac{h_2}{k_2}$ *are adjacent Farey fractions in the Farey sequence of order* N, *with* $N := \lfloor n^{\frac{1}{2}} \rfloor$. *Then*

$$z^{\frac{1}{2}} \cdot I_{a,c,k,\nu}(z) \ll k \cdot n^{\frac{1}{4}} \cdot g_{a,c,k,\nu},$$

where $g_{a,c,k,\nu} := \left(\min\left(6kc\left\{\frac{\nu}{k} - \frac{1}{6k} + \frac{a}{c}\right\}, 6kc\left\{\frac{\nu}{k} - \frac{1}{6k} - \frac{a}{c}\right\}\right)\right)^{-1}$, *where* $\{x\} := x - \lfloor x \rfloor$ *for* $x \in \mathbb{R}$. *Here the implied constant is independent of* a, k, *and* ν.

Proof. We write $\frac{\pi z}{k} = Ce^{iA}$ with $C > 0$. Then $|A| < \frac{\pi}{2}$ since $\text{Re}(z) > 0$. Making the substitution $\tau = \frac{\pi z x}{k}$ gives

$$(3.1) \qquad z^{\frac{1}{2}} \cdot I_{a,c,k,\nu}(z) = \frac{k}{\pi z^{\frac{1}{2}}} \int_S e^{-\frac{3k\tau^2}{\pi z}} \cdot H_{a,c}\left(\frac{\pi i \nu}{k} - \frac{\pi i}{6k} - \tau\right) d\tau,$$

where τ runs on the ray through 0 of elements with argument $\pm A$. One can see that for $0 \leq t \leq A$

$$(3.2) \qquad \left| e^{-\frac{3kR^2 e^{2it}}{\pi z}} \cdot H_{a,c}\left(\frac{\pi i \nu}{k} - \frac{\pi i}{6k} \pm R e^{it}\right) \right| \to 0 \qquad (R \to \infty).$$

Moreover, since c is odd, the integrant in (3.1) can only have poles in points ir with $r \in \mathbb{R}\backslash\{0\}$. Thus we can shift the path of integration to the real line and get

$$z^{\frac{1}{2}} \cdot I_{a,c,k,\nu}(z) = \frac{k}{\pi z^{\frac{1}{2}}} \int_{\mathbb{R}} e^{-\frac{3kt^2}{\pi z}} \cdot H_{a,c}\left(\frac{\pi i \nu}{k} - \frac{\pi i}{6k} - t\right) dt.$$

We can show the following estimates

$$\left| e^{-\frac{3kt^2}{\pi z}} \right| = e^{-\frac{3k}{\pi}\mathrm{Re}\left(\frac{1}{z}\right)t^2},$$

$$\left| \cosh\left(\frac{\pi i \nu}{k} - \frac{\pi i}{6k} - t\right) \right| \leq e^t,$$

$$\left| \sinh\left(\frac{\pi i \nu}{k} - \frac{\pi i}{6k} - t \pm \frac{\pi i a}{c}\right) \right| \geq \begin{cases} \frac{e^t}{2\sqrt{2}} & \text{if } t \geq 1, \\ \left| \sin\left(\frac{\pi \nu}{k} - \frac{\pi}{6k} \pm \frac{\pi a}{c}\right) \right| & \text{if } t \leq 1, \end{cases}$$

$$\left| \sin\left(\frac{\pi \nu}{k} - \frac{\pi}{6k} + \frac{\pi a}{c}\right) \right| \left| \sin\left(\frac{\pi \nu}{k} - \frac{\pi}{6k} - \frac{\pi a}{c}\right) \right|$$
$$\gg \left(\min\left(\left\{\frac{\nu}{k} - \frac{1}{6k} + \frac{a}{c}\right\}, \left\{\frac{\nu}{k} - \frac{1}{6k} - \frac{a}{c}\right\}\right) \right).$$

Thus

$$z^{\frac{1}{2}} \cdot I_{a,c,k,\nu}(z) \ll \frac{k}{\left(\min\left(\left\{\frac{\nu}{k} - \frac{1}{6k} + \frac{a}{c}\right\}, \left\{\frac{\nu}{k} - \frac{1}{6k} - \frac{a}{c}\right\}\right)\right)|z|^{\frac{1}{2}}} \int_{\mathbb{R}} e^{-\frac{3k}{\pi}t^2 \mathrm{Re}\left(\frac{1}{z}\right)} dt.$$

Making the substitution $t \mapsto \sqrt{\frac{3k\mathrm{Re}\left(\frac{1}{z}\right)}{\pi}} \cdot t$ and using the estimate

$$\mathrm{Re}\left(\frac{1}{z}\right)^{-\frac{1}{2}} \cdot |z|^{-\frac{1}{2}} \leq 2^{\frac{1}{4}} \cdot n^{\frac{1}{4}} \cdot k^{-\frac{1}{2}}$$

gives the lemma. $\qquad \square$

We next estimate certain sums of Kloosterman type.

Lemma 3.2. *Let $n, m \in \mathbb{Z}$, $0 \leq \sigma_1 < \sigma_2 \leq k$, $D \in \mathbb{Z}$ with $(D, k) = 1$.*

(1) *We have*

$$(3.3) \qquad \sum_{\substack{h \pmod{k}^* \\ \sigma_1 \leq Dh' \leq \sigma_2}} \omega_{h,k} \cdot e^{\frac{2\pi i}{k}(hn + h'm)} \ll \gcd(24n + 1, k)^{\frac{1}{2}} \cdot k^{\frac{1}{2}+\epsilon}.$$

(2) *If $c \mid k$, then we have*

(3.4)

$$(-1)^{ak+1} \sin\left(\frac{\pi a}{c}\right) \sum_{\substack{h \pmod{k}^* \\ \sigma_1 \le Dh' \le \sigma_2}} \frac{\omega_{h,k}}{\sin\left(\frac{\pi a h'}{c}\right)} \cdot e^{-\frac{3\pi i a^2 k_1 h'}{c}} \cdot e^{\frac{2\pi i}{k}(hn+h'm)} \ll \gcd(24n+1,k)^{\frac{1}{2}} \cdot k^{\frac{1}{2}+\epsilon},$$

where the implied constants are independent of a and k.

Proof. Equation (3.3) is basically proven in [2], thus we only consider (3.4). We set $\tilde{c} := c$ if k is odd and $\tilde{c} := 2c$ if k is even. Clearly $\frac{e^{-\frac{3\pi i a^2 k_1 h'}{c}}}{\sin\left(\frac{\pi a h'}{c}\right)}$ only depends on the residue class of h' (mod $\tilde{c}$). Thus we can write (3.4) as

$$(-1)^{ak+1} \sin\left(\frac{\pi a}{c}\right) \sum_{c_j} \frac{e^{-\frac{3\pi i a^2 k_1 c_j}{c}}}{\sin\left(\frac{\pi a c_j}{c}\right)} \sum_{\substack{h \pmod{k}^* \\ \sigma_1 \le Dh' \le \sigma_2 \\ h' \equiv c_j \pmod{\tilde{c}}}} \omega_{h,k} \cdot e^{\frac{2\pi i}{k}(hn+h'm)},$$

where c_j runs through a set of primitive residues (mod $\tilde{c}$). The inner sum can be rewritten as

$$\frac{1}{\tilde{c}} \sum_{\substack{h \pmod{k}^* \\ \sigma_1 \le Dh' \le \sigma_2}} \omega_{h,k} \cdot e^{\frac{2\pi i}{k}(hn+h'm)} \sum_{r \pmod{\tilde{c}}} e^{\frac{2\pi i r}{\tilde{c}}(h'-c_j)}$$

$$= \frac{1}{\tilde{c}} \sum_{r \pmod{\tilde{c}}} e^{-\frac{2\pi i r c_j}{\tilde{c}}} \sum_{\substack{h \pmod{k}^* \\ \sigma_1 \le Dh' \le \sigma_2}} \omega_{h,k} \cdot e^{\frac{2\pi i}{k}\left(hn+h'\left(m+\frac{kr}{\tilde{c}}\right)\right)}.$$

Using (3.3) now easily gives (3.4). □

Remark. If $a = 1$ and $c = 3$, then it is not hard to see that the left hand side of (3.4) simplifies to

$$\sum_{\substack{h \pmod{k}^* \\ \sigma_1 \le Dh' \le \sigma_2}} \left(\frac{h}{3}\right) \cdot \omega_{h,k} \cdot e^{\frac{2\pi i}{k}(hn+h'm)}.$$

4. Proof of Theorem 1.1

Proof of Theorem 1.1. For the proof of Theorem 1.1 we use the Hardy-Ramanujan method. By Cauchy's Theorem we have for $n > 0$

$$A\left(\frac{a}{c}; n\right) = \frac{1}{2\pi i} \int_C \frac{N\left(\frac{a}{c}; q\right)}{q^{n+1}} \, dq,$$

where C is an arbitrary path inside the unit circle surrounding 0 counterclockwise. Choosing the circle with radius $e^{-\frac{2\pi}{n}}$ and as a parametrisation $q = e^{-\frac{2\pi}{n}+2\pi i t}$ with $0 \le t \le 1$, gives

$$A\left(\frac{a}{c}; n\right) = \int_0^1 N\left(\frac{a}{c}; e^{-\frac{2\pi}{n}+2\pi i t}\right) \cdot e^{2\pi - 2\pi i n t} \, dt.$$

Define

$$\vartheta'_{h,k} := \frac{1}{k(k_1 + k)}, \quad \vartheta''_{h,k} := \frac{1}{k(k_2 + k)},$$

where $\frac{h_1}{k_1} < \frac{h}{k} < \frac{h_2}{k_2}$ are adjacent Farey fractions in the Farey sequence of order $N := \lfloor n^{1/2} \rfloor$. From the theory of Farey fractions it is known that

$$(4.1) \qquad \frac{1}{k + k_j} \le \frac{1}{N + 1} \qquad (j = 1, 2).$$

We decompose the path of integration in paths along the Farey arcs $-\vartheta'_{h,k} \le \Phi \le \vartheta''_{h,k}$, where $\Phi = t - \frac{h}{k}$ and $0 \le h \le k \le N$ with $(h, k) = 1$. Thus

$$A\left(\frac{a}{c}; n\right) = \sum_{h,k} e^{-\frac{2\pi i h n}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} N\left(\frac{a}{c}; e^{\frac{2\pi i}{k}(h + iz)}\right) \cdot e^{\frac{2\pi n z}{k}} \, d\Phi,$$

where $z = \frac{k}{n} - k\Phi i$. Applying Theorem 2.1 gives

$$A\left(\frac{a}{c}; n\right) = i \sin\left(\frac{\pi a}{c}\right) \sum_{\substack{h,k \\ c|k}} \omega_{h,k} \frac{(-1)^{ak+1}}{\sin\left(\frac{\pi a h'}{c}\right)} \cdot e^{-\frac{3\pi i a^2 k_1 h'}{c} - \frac{2\pi i h n}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{-\frac{1}{2}} \cdot e^{\frac{2\pi z}{k}\left(n - \frac{1}{24}\right) + \frac{\pi}{12kz}}$$

$$\times N\left(\frac{ah'}{c}; q_1\right) d\Phi - 4i \sin\left(\frac{\pi a}{c}\right) \sum_{\substack{h,k \\ c\nmid k}} \omega_{h,k} (-1)^{ak+l} e^{-\frac{2\pi i h' sa}{c} - \frac{3\pi i h' a^2 k_1}{cc_1} + \frac{6\pi i h' la}{cc_1} - \frac{2\pi i h n}{k}}$$

$$\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{-\frac{1}{2}} \cdot e^{\frac{2\pi z}{k}\left(n - \frac{1}{24}\right) + \frac{\pi}{12kz}} \cdot q_1^{\frac{sl}{c_1} - \frac{3l^2}{2c_1^2}} \cdot N\left(ah', \frac{lc}{c_1}, c; q_1\right) d\Phi + 2\sin^2\left(\frac{\pi a}{c}\right) \sum_{h,k} \frac{\omega_{h,k}}{k} \cdot e^{-\frac{2\pi i h n}{k}}$$

$$\sum_{\nu \pmod{k}} (-1)^\nu e^{-\frac{3\pi i h' \nu^2}{k} + \frac{\pi i h' \nu}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} e^{\frac{2\pi z}{k}\left(n - \frac{1}{24}\right)} \cdot z^{\frac{1}{2}} \cdot I_{a,c,k,\nu}(z) d\Phi =: \sum_1 + \sum_2 + \sum_3 .$$

To estimate $\sum_1$, we write

$$N\left(\frac{ah'}{c}; q_1\right) =: 1 + \sum_{r \in \mathbb{N}} a(r) \cdot e^{\frac{2\pi i m_r h'}{k}} \cdot e^{-\frac{2\pi r}{kz}},$$

where m_r is a sequence in $\mathbb{Z}$ and the coefficients $a(r)$ are independent of $a, c, k,$ and h. We treat the constant term and the term coming from from $r \ge 1$ seperately since they contribute to the main term and to the error term, respectively. We denote the associated sums by S_1 and S_2, respectively and first estimate S_2. Throughout we need the easily verified fact that $\mathrm{Re}(z) = \frac{k}{n}$, $\mathrm{Re}\left(\frac{1}{z}\right) > \frac{k}{2}$, $|z|^{-\frac{1}{2}} \le n^{\frac{1}{2}} \cdot k^{-\frac{1}{2}}$, and $\vartheta'_{h,k} + \vartheta''_{h,k} \le \frac{2}{k(N+1)}$. Since $k_1, k_2 \le N$, we can write

$$(4.2) \qquad \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} = \int_{-\frac{1}{k(N+k)}}^{\frac{1}{k(N+k)}} + \int_{-\frac{1}{k(k_1+k)}}^{-\frac{1}{k(N+k)}} + \int_{\frac{1}{k(N+k)}}^{\frac{1}{k(k_2+k)}}$$

and denote the associated sums by S_{21}, S_{22}, and S_{23}, respectively.

We first consider S_{21}. Using Lemma 3.2 gives

$$
S_{21} \ll \left| \sum_{r=1}^{\infty} a(r) \sum_{c|k} (-1)^{ak+1} \sin\left(\frac{\pi a}{c}\right) \sum_{h} \frac{\omega_{h,k}}{\sin\left(\frac{\pi ah'}{c}\right)} \cdot e^{-\frac{3\pi i a^2 k_1 h'}{c} - \frac{2\pi i h n}{k} + \frac{2\pi i m_r h'}{k}} \right.
$$

$$
\left. \int_{-\frac{1}{k(N+k)}}^{\frac{1}{k(N+k)}} z^{-\frac{1}{2}} \cdot e^{-\frac{2\pi}{kz}\left(r-\frac{1}{24}\right)+\frac{2\pi z}{k}\left(n-\frac{1}{24}\right)} d\Phi \right| \ll \sum_{r=1}^{\infty} |a(r)| \cdot e^{-\pi r} \sum_{k} k^{-1+\epsilon} \cdot (24n-1,k)^{\frac{1}{2}}
$$

$$
\ll \sum_{\substack{d|(24n-1) \\ d \leq N}} d^{\frac{1}{2}} \sum_{k \leq \frac{N}{d}} (dk)^{-1+\epsilon} \ll n^{\epsilon} \sum_{\substack{d|(24n-1) \\ d \leq N}} d^{-\frac{1}{2}} \ll n^{\epsilon}.
$$

Since S_{22} and S_{23} are treated in exactly the same way we only consider S_{22}. Writing

$$
\int_{-\frac{1}{k(k+k_1)}}^{-\frac{1}{k(N+k)}} = \sum_{l=k_1+k}^{N+k-1} \int_{-\frac{1}{kl}}^{-\frac{1}{k(l+1)}},
$$

we see

$$
(4.3) \quad S_{22} \ll \left| \sum_{r=1}^{\infty} a(r) \sum_{c|k} \sum_{l=N+1}^{N+k-1} \int_{-\frac{1}{kl}}^{-\frac{1}{k(l+1)}} z^{-\frac{1}{2}} \cdot e^{-\frac{2\pi}{kz}\left(r-\frac{1}{24}\right)+\frac{2\pi z}{k}\left(n-\frac{1}{24}\right)} d\Phi \right.
$$

$$
\left. (-1)^{ak+1} \sin\left(\frac{\pi a}{c}\right) \sum_{\substack{h \\ N < k+k_1 \leq l}} \frac{\omega_{h,k}}{\sin\left(\frac{\pi ah'}{c}\right)} \cdot e^{-\frac{3\pi i a^2 k_1 h'}{c}} \cdot e^{-\frac{2\pi i h n}{k}} e^{\frac{2\pi i m_r h'}{k}} \right|.
$$

It follows from the theory of Farey fractions that

$$
k_1 \equiv -h' \pmod{k}, \quad k_2 \equiv h' \pmod{k},
$$
$$
N - k < k_1 \leq N, \quad N - k < k_2 \leq N.
$$

Thus we can use Lemma 3.2 and estimate (4.3) as in the case of S_{21}. Therefore $\sum_1$ equals

$$
i \sin\left(\frac{\pi a}{c}\right) \sum_{\substack{h,k \\ c|k}} \omega_{h,k} \cdot \frac{(-1)^{ak+1}}{\sin\left(\frac{\pi ah'}{c}\right)} \cdot e^{-\frac{3\pi i a^2 k_1 h'}{c} - \frac{2\pi i h n}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{-\frac{1}{2}} \cdot e^{\frac{2\pi z}{k}\left(n-\frac{1}{24}\right)+\frac{\pi}{12kz}} d\Phi + O\left(n^{\epsilon}\right).
$$

The sum $\sum_2$ is treated in a similar manner, we make some comments about necessary modifications. One can show that one can write

(4.4)

$$
e^{-\frac{2\pi i h' sa}{c} - \frac{3\pi i h' a^2 k_1}{cc_1} + \frac{6\pi i h' la}{cc_1} + \frac{\pi}{12kz}} \cdot q_1^{\frac{sl}{c_1} - \frac{3l^2}{2c_1^2}} \cdot N\left(ah', \frac{lc}{c_1}, c; q_1\right) =: \sum_{r \geq r_0} b(r) e^{\frac{2\pi i m_r h'}{k}} e^{-\frac{\pi i r}{12kc^2 z}},
$$

where m_r is a sequence in $\mathbb{Z}$, $r_0 \in \mathbb{Z}$. A lengthy but straightforward calculation shows that the terms with r negative, i.e., the the part that gives a contribution for the main term, can only arise if $s \in \{0, 3\}$. If $s = 0$, then the contribution is given by

$$\frac{i}{2} e^{-\frac{3\pi i h' a^2 k_1}{cc_1} + \frac{6\pi i h' la}{cc_1} - \frac{\pi i a h'}{c} + \frac{\pi}{12kz}} \cdot q_1^{-\frac{3l^2}{2c_1^2} + \frac{l}{2c_1}} \sum_{\substack{r \\ \delta_{c,k,r} > 0}} e\left(-\frac{ah'r}{c}\right) \cdot q_1^{\frac{lr}{c_1}}.$$

If $s = 3$, then the contribution is given by

$$\frac{i}{2} e^{-\frac{6\pi i a h' a}{c} - \frac{3\pi i h' a^2 k_1}{cc_1} + \frac{6\pi i h' la}{cc_1} + \frac{\pi i a h'}{c} + \frac{\pi}{12kz}} \cdot q_1^{\frac{5l}{2c_1} - \frac{3l^2}{2c_1^2} - 1} \sum_{\substack{r \\ \delta_{c,k,r} > 0}} e\left(\frac{ah'r}{c}\right) \cdot q_1^{\left(1 - \frac{l}{c_1}\right)r}.$$

One can write the sum over k in groups in which the k's have the same values for c_1 and l and thus the same values for $\delta_{c,k,r}$, where $\delta_{c,k,r}$ was defined in (1.13). In each class the condition $\delta_{c,k,r} > 0$ is now independent of the k in this class; in every group there are only finitely many terms with $\delta_{c,k,r} > 0$, and the number is bounded in terms of c. Moreover the coefficient $b(r)$ is independent of k and a in a fixed class. Now the terms with negative exponents can be estimated as before, thus $\sum_2$ equals

$$2\sin\left(\frac{\pi a}{c}\right) \sum_{\substack{k,r \\ c\nmid k \\ \delta_{c,k,r} > 0}} (-1)^{ak+l} \sum_h \omega_{h,k} e^{\frac{2\pi i}{k}(-nh + m_{a,c,k,r}h')} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} z^{-\frac{1}{2}} e^{\frac{2\pi z}{k}\left(n - \frac{1}{24}\right) + \frac{2\pi}{kz}\delta_{c,k,r}} d\Phi + O\left(n^\epsilon\right).$$

We turn back to the estimation of $\sum_1$. We can write

$$\int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} = \int_{-\frac{1}{kN}}^{\frac{1}{kN}} - \int_{-\frac{1}{kN}}^{-\frac{1}{k(k+k_1)}} - \int_{\frac{1}{k(k+k_2)}}^{\frac{1}{kN}}$$

and denote the associated sums by S_{11}, S_{12}, and S_{13}, respectively. The sums S_{12} and S_{13} contribute to the error terms. Since they have the same shape, we only consider S_{12}. Writing

$$\int_{-\frac{1}{kN}}^{-\frac{1}{k(k+k_1)}} = \sum_{l=N}^{k+k_1-1} \int_{-\frac{1}{kl}}^{-\frac{1}{k(l+1)}}$$

gives

$$S_{12} \ll \sum_{c|k} \sum_{l=N}^{N+k-1} \int_{-\frac{1}{kl}}^{-\frac{1}{k(l+1)}} z^{-\frac{1}{2}} e^{\frac{\pi}{12kz} + \frac{2\pi z}{k}\left(n - \frac{1}{24}\right)} d\Phi$$

$$(-1)^{ak+1} \sin\left(\frac{\pi a}{c}\right) \sum_{\substack{h \\ l < k+k_1-1 \le N+k-1}} \frac{\omega_{h,k}}{\sin\left(\frac{\pi a h'}{c}\right)} \cdot e^{-\frac{3\pi i a^2 k_1 h'}{c}} \cdot e^{-\frac{2\pi i h n}{k}}.$$

Using $\operatorname{Re}(z) = \frac{k}{n}$, $\operatorname{Re}\left(\frac{1}{z}\right) < k$, and $|z|^2 \geq \frac{k^2}{n^2}$, this sum can be estimated as before against $O\left(n^\epsilon\right)$. Thus, using (1.11),

$$\sum_1 = i \sum_{c|k} B_{a,c,k}(-n,0) \int_{-\frac{1}{kN}}^{\frac{1}{kN}} z^{-\frac{1}{2}} \cdot e^{\frac{2\pi z}{k}\left(n - \frac{1}{24}\right) + \frac{\pi}{12kz}} \, d\Phi + O\left(n^\epsilon\right).$$

In the same way, we obtain, using (1.12),

$$\sum_2 = 2\sin\left(\frac{\pi a}{c}\right) \sum_{\substack{k,r \\ c\nmid k \\ \delta_{c,k,r} > 0}} D_{a,c,k}(-n, m_{a,c,k,r}) \int_{-\frac{1}{kN}}^{\frac{1}{kN}} z^{-\frac{1}{2}} \cdot e^{\frac{2\pi z}{k}\left(n - \frac{1}{24}\right) + \frac{2\pi}{kz}\delta_{c,k,r}} \, d\Phi + O\left(n^\epsilon\right).$$

To finish the estimation of $\sum_1$ and $\sum_2$ we have to consider integrals of the form

$$I_{k,r} := \int_{-\frac{1}{kN}}^{\frac{1}{kN}} z^{-\frac{1}{2}} \cdot e^{\frac{2\pi}{k}\left(z\left(n - \frac{1}{24}\right) + \frac{r}{z}\right)} d\Phi.$$

Substituting $z = \frac{k}{n} - ik\Phi$ gives

$$(4.5) \qquad I_{k,r} = \frac{1}{ki} \int_{\frac{k}{n} - \frac{i}{N}}^{\frac{k}{n} + \frac{i}{N}} z^{-\frac{1}{2}} \cdot e^{\frac{2\pi}{k}\left(z\left(n - \frac{1}{24}\right) + \frac{r}{z}\right)} \, dz.$$

We now denote the circle through $\frac{k}{n} \pm \frac{i}{N}$ and tangent to the imaginary axis at 0 by Γ. If $z = x + iy$, then Γ is given by $x^2 + y^2 = \alpha x$, with $\alpha = \frac{k}{n} + \frac{n}{N^2 k}$. Using the fact that $2 > \alpha > \frac{1}{k}$, $\operatorname{Re}(z) \leq \frac{k}{n}$, and $\operatorname{Re}\left(\frac{1}{z}\right) < k$ on the smaller arc we can show that the integral along the smaller arc is in $O\left(n^{-\frac{3}{4}}\right)$. Moreover the path of integration in (4.5) can be changed by Cauchy's Theorem into the larger arc of Γ. Thus

$$I_{k,r} = \frac{1}{ki} \int_\Gamma z^{-\frac{1}{2}} \cdot e^{\frac{2\pi}{k}\left(z\left(n - \frac{1}{24}\right) + \frac{r}{z}\right)} \, dz + O\left(n^{-\frac{3}{4}}\right).$$

Making the substitution $t = \frac{2\pi r}{kz}$ gives

$$I_{k,r} = \frac{2\pi}{k} \left(\frac{2\pi r}{k}\right)^{\frac{1}{2}} \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} t^{-\frac{3}{2}} \cdot e^{t + \frac{\alpha}{t}} \, dt + O\left(n^{-\frac{3}{4}}\right),$$

where $\gamma \in \mathbb{R}$ and $\alpha = \frac{\pi^2 r}{6k^2}(24n - 1)$. By the Hankel integral formula we now get

$$I_{k,r} = \frac{4\sqrt{3}}{\sqrt{k\left(24n - 1\right)}} \sinh\left(\sqrt{\frac{2r(24n - 1)}{3}} \frac{\pi}{k}\right) + O\left(n^{-\frac{3}{4}}\right).$$

Thus we obtain, estimating the Kloosterman sums with the same arguments as before and using Lemma 3.1,

$$\sum_1 + \sum_2 = \frac{4\sqrt{3}i}{\sqrt{24n-1}} \sum_{c|k} \frac{B_{a,c,k}(-n,0)}{\sqrt{k}} \cdot \sinh\left(\frac{\pi}{6k}\sqrt{24n-1}\right)$$

$$+ \frac{8\sqrt{3}\sin\left(\frac{\pi a}{c}\right)}{\sqrt{24n-1}} \sum_{\substack{k,r \\ c\nmid k \\ \delta_{c,k,r}>0}} \frac{D_{a,c,k}(-n,m_{a,c,k,r})}{\sqrt{k}} \cdot \sinh\left(\sqrt{\frac{2\delta_{c,k,r}(24n-1)}{3}}\frac{\pi}{k}\right) + O\left(n^\epsilon\right).$$

To estimate $\sum_3$, we split the path of integration as in (4.2) and estimate the sums in the same way as before, using that

$$\sum_{1\le\nu\le k} g_{a,c,k,\nu} \ll \sum_{1\le\nu\le 6ck} \frac{1}{\nu} \ll k^\epsilon.$$

$\square$

Proof of Corollary 1.2. Corollary 1.2 follows directly from the identity

$$(4.6) \qquad \sum_{n=0}^{\infty} N(a,c;n)q^n = \frac{1}{c}\sum_{n=0}^{\infty} p(n)q^n + \frac{1}{c}\sum_{j=1}^{c-1} \zeta_c^{-aj} \cdot R(\zeta_c^j; q)$$

and (1.4). $\square$

5. Proof of Theorem 1.3

Here we prove the Andrews-Lewis conjecture. Using equation (4.6) and

$$N\left(\frac{1}{3};q\right) = N\left(\frac{2}{3};q\right)$$

easily gives that

$$N(0,3;n) < N(1,3;n) \Leftrightarrow A\left(\frac{1}{3};n\right) < 0.$$

Thus Theorem 2 follows from the following.

Proposition 5.1. *We have for all $n \notin \{1,3,7\}$*

$$(5.1) \qquad\qquad A\left(\frac{1}{3};3n\right) \quad < \quad 0,$$

$$(5.2) \qquad\qquad A\left(\frac{1}{3};3n+1\right) \quad > \quad 0,$$

$$(5.3) \qquad\qquad A\left(\frac{1}{3};3n+2\right) \quad < \quad 0.$$

Remark. For $n \in \{1,3,7\}$, equation (5.2) and (5.3) still hold and in (5.1) we have equality.

Proof. With MAPLE we compute that the proposition holds for $n \le 3000$. Thus in the following, we may assume that $n > 3000$. Actually we only need that $n \ge 300$.

We first consider the main term given by

$$(5.4) \qquad \frac{4\sqrt{3}i}{(24n-1)^{\frac{1}{2}}} \sum_{3|k} \frac{B_{1,3,k}(-n,0)}{\sqrt{k}} \cdot \sinh\left(\frac{\pi}{6k}\sqrt{24n-1}\right).$$

Using definition (1.10) it is easy to see that

$$B_{1,3,3}(-n,0) = 2i\sin\left(\frac{\pi}{18} - \frac{2\pi n}{3}\right).$$

Thus the term corresponding to $k = 3$ is given by

$$-\frac{8\sin\left(\frac{\pi}{18} - \frac{2\pi n}{3}\right) \cdot \sinh\left(\frac{\pi}{18}\sqrt{24n-1}\right)}{(24n-1)^{\frac{1}{2}}}.$$

Using the trivial bound for Kloosterman sums gives that the remaining terms can be estimated against

$$\frac{12}{(24n-1)^{\frac{1}{2}}} \sum_{2 \le k \le \frac{N}{3}} k^{\frac{1}{2}} \cdot \sinh\left(\frac{\pi}{18k}\sqrt{24n-1}\right).$$

Next we make the O-term in Theorem 1.1 explicit. For this we use the same notation as in the proof of Theorem 1.1. Instead of using Lemma 3.2, we estimate the Kloosterman sums trivially. The contribution coming from $N\left(\frac{h'}{3}; q_1\right)$ can be estimated against

$$\frac{2 \cdot e^{2\pi + \frac{\pi}{24}}}{\sqrt{3}} \sum_{r=1}^{\infty} |a(r)| \cdot e^{-\pi r} \sum_{1 \le k \le \frac{N}{3}} k^{-\frac{1}{2}}.$$

Computing the first few coefficients and then using that for $n > 1$

$$p(n) < \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

easily gives that

$$\sum_{r=1}^{\infty} |a(r)| \cdot e^{-\pi r} \le 0.06.$$

In the same way we can estimate the contribution coming from $N(h',l,3;q_1)$ by

$$4\sqrt{3} \cdot e^{2\pi} \sum_{r=1}^{\infty} |b(r)| \cdot e^{-\frac{\pi r}{144}} \sum_{\substack{1 \le k \le N \\ 3 \nmid k}} k^{-\frac{1}{2}}.$$

Moreover, distinguishing the cases $l = 1$ and $l = 2$, we get

$$\sum_{r=1}^{\infty} |b(r)| \cdot e^{-\frac{\pi r}{216}} \le 0.353.$$

By making the path of integration symmetric, we introduce an error that can be estimated against

$$2\sqrt{3} \cdot e^{2\pi + \frac{\pi}{12}} \cdot n^{-\frac{1}{2}} \sum_{1 \leq k \leq \frac{N}{3}} k^{\frac{1}{2}}.$$

Integrating along the smaller arc of Γ gives an error that can be estimated against

$$8\pi \cdot e^{2\pi + \frac{\pi}{24}} \cdot n^{-\frac{3}{4}} \sum_{1 \leq k \leq \frac{N}{3}} k.$$

Moreover the estimate in Lemma 3.1 can be made explicit as

$$2^{\frac{5}{4}} \cdot \frac{e + e^{-1}}{3} \cdot n^{\frac{1}{4}} \left(\min \left(\left\{ \frac{\nu}{k} - \frac{1}{6k} + \frac{1}{3} \right\}, \left\{ \frac{\nu}{k} - \frac{1}{6k} + \frac{1}{3} \right\} \right) \right)^{-1}.$$

This easily gives

$$\sum_{3} \leq 2^{\frac{1}{4}} \cdot (e + e^{-1}) \cdot e^{2\pi} \cdot n^{-\frac{1}{4}} \sum_{k} \frac{1}{k} \sum_{\nu=1}^{k} \left(\min \left(\left\{ \frac{\nu}{k} - \frac{1}{6k} + \frac{1}{3} \right\}, \left\{ \frac{\nu}{k} - \frac{1}{6k} - \frac{1}{3} \right\} \right) \right)^{-1}.$$

Combining all errors one can show that the term in (5.4) is dominant for $n \geq 300$. $\qquad \square$

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